

Another proof of the Semistable Reduction Theorem

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Abstract

We give a new proof of the Semistable Reduction Theorem for curves. The main idea is to present a curve Y over a local field K as a finite cover of the projective line $X = \mathbb{P}_K^1$. By successive blowups (and after replacing K by a suitable finite extension) we construct a semistable model of X whose normalization with respect to the cover is a semistable model of Y .

1 Introduction

1.1 Let K be a field which is complete with respect to a discrete valuation v . We let R denote the valuation ring of v , $\mathfrak{m} \triangleleft R$ the maximal ideal of R and $k := R/\mathfrak{m}$ the residue field.

Let X be a smooth projective and absolutely irreducible curve over K . A *model* of X is a normal, flat and proper R -scheme X_R such that $X_R \otimes_R K = X$. Given a model X_R of X , its special fiber is denoted by $X_s := X_R \otimes_R k$. The k -scheme X_s is proper, connected and of pure dimension one. We say that X_R is a *semistable model* of X if X_s is *nodal*, i.e. all singular points are ordinary double points. We say that X has *semistable reduction* if it has a semistable model X_R .

Theorem 1.1 (Semistable Reduction Theorem) *There exists a finite extension L/K such that the curve $X_L := X \otimes_K L$ has semistable reduction (w.r.t. the unique extension of v to L).*

The first proof of this theorem was given by Deligne and Mumford ([11], Corollary 2.7). Since then, many more proofs have appeared in the literature, see e.g. [1].

1.2 The question that originally motivated the present paper is: how can one explicitly determine a semistable model of a given curve? In a way the proof of Theorem 1.1 by Deligne and Mumford is constructive: choose $n \geq 3$ prime to the residue characteristic of K and let L/K be the smallest field extension over which the n -torsion points of the jacobian of X become rational. Then the minimal regular model of X_L is semistable. In theory, this gives an algorithm

to determine a semistable model. It seems, however, that several steps in this algorithm are today still computationally too expensive to be practical for curves of genus $g \geq 3$.

In this paper we work out a new proof of Theorem 1.1 which we hope will ultimately lead to a more practical algorithm. The starting point of our investigation was a paper by M. Matignon ([16], see also [14]), which gives an algorithm to compute the semistable reduction of p -cyclic covers of the projective line (satisfying an additional assumption). Trying to generalize Matignon's method to a more general situation, we noticed that it could be used as a germ for a new proof of Theorem 1.1.

1.3 Let us give a brief sketch of our proof. The first idea is to view the curve under consideration as a finite cover of the projective line. So we start with a smooth projective K -curve Y and choose a nonconstant separable finite morphism $\phi : Y \rightarrow X := \mathbb{P}_K^1$. For any model X_R of X we obtain a model Y_R of Y and a finite R -morphism $\phi_R : Y_R \rightarrow X_R$ by normalization of X_R in Y . The goal is now to determine a semistable model X_R of X such that Y_R is semistable as well. We show that this is possible after replacing K by a finite extension (Theorem 2.10) and obtain Theorem 1.1 as an immediate consequence.

In order to prove Theorem 2.10, we may assume that the cover ϕ is Galois. Let G denote the Galois group of the cover ϕ . If the order of G is prime to the residue characteristic of K , then it is well known how to obtain a model X_R with the desired properties: it suffices to take a semistable model which separates the branch points of the cover ϕ (see e.g. [15], §10.4). In particular, if the residue characteristic is zero, then the Semistable Reduction Theorem is relatively easy to prove.

In the general case, let X_R be any semistable model of $X = \mathbb{P}_K^1$. Let Y_R be the normalization of X_R in Y . By a theorem of Epp ([12]) we may assume that the special fiber Y_s of Y_R is reduced. If Y_R is semistable then we are done. Otherwise, there exists a singular point $y \in Y_s$ which is not an ordinary double point. Let $x \in X_s$ denote the image of y under the map ϕ_R . The crucial step of our proof is to show that there exists a blowup $f : X'_R \rightarrow X_R$ with center x which 'improves the situation'. To make this a bit more precise, let Y'_R denote the normalization of X'_R in Y . Then the induced map $g : Y'_R \rightarrow Y_R$ is a blowup with center $\phi_R^{-1}(x)$. We say that X'_R is an *improvement* of X_R at y if the singularities of the special fiber of Y'_R which lie on the fiber $g^{-1}(y)$ are 'less bad' than the singularity $y \in Y_s$ ('badness' of singularities can be measured by a suitable numerical invariant). Once the existence of an improvement has been shown, the proof of Theorem 2.10 is straightforward: start with some semistable model X_R of X (e.g. the smooth model \mathbb{P}_R^1) and repeatedly apply the above improvement procedure. After a finite number of steps we obtain a model X'_R whose normalization in Y is a semistable model of Y .

Our proof that an improvement exists is local in the sense that it depends only on the formal completions of X_R at x and of Y_R at y . Instead of working with formal schemes, the crucial step of the proof is phrased in the language of

rigid geometry, as follows.

Let X^{rig} and Y^{rig} denote the rigid analytic spaces associated to the K -curves X and Y . We consider the formal fiber $X_x :=]x[_{X_R} \subset X^{\text{rig}}$ of the point x , i.e. the subset of points on X^{rig} which specialize to x , and likewise the formal fiber $Y_y :=]y[_{Y_R} \subset Y^{\text{rig}}$ of y . Then ϕ induces a finite Galois cover $\phi_y : Y_y \rightarrow X_x$ of smooth rigid analytic spaces of dimension one. The fact that x is a smooth point of the special fiber X_s implies that X_x is an open disk, i.e. isomorphic to the rigid space

$$\{t \in \mathbb{A}_K^1 \mid |t| < 1\}.$$

Let $D \subset X_x$ be an affinoid disk. By this we mean that after a finite extension of K there exists a parameter t as above and some $\epsilon \in \sqrt{|K^\times|}$, with $0 < \epsilon < 1$, such that D is the open subspace of X_x defined by the condition $|t| \leq \epsilon$. To D one can associate a blowup $f : X'_R \rightarrow X_R$ with center x whose exceptional divisor is a (-1) -curve. The blowup $g : Y'_R \rightarrow Y_R$ induced by f is associated to the affinoid subdomain $U := \phi_y^{-1}(D) \subset Y_y$.

We say that the affinoid disk $D \subset X_x$ is *exhausting* if the complement $Y_y - \phi_y^{-1}(D)$ decomposes as a union of open annuli. Let \mathcal{D} denote the set of all exhausting disks. A well known lemma (see e.g. [7], Lemma 2.4) says that every sufficiently large affinoid disk $D \subset X_s$ is exhausting. In particular, \mathcal{D} is nonempty. One easily shows:

Proposition: *The blowup $f : X'_R \rightarrow X_R$ associated to D is an improvement at y if and only if D is a minimal element of \mathcal{D} (with respect to inclusion).*

Therefore, we have reduced the proof of the Semistable Reduction Theorem to the claim that the set \mathcal{D} has a minimal element.¹

Our proof of the existence of a minimal exhausting disk is divided into two cases. We first assume that the Galois group G of the cover is solvable. In this case the proof can be easily reduced to the case that G is cyclic of prime order. Under the latter assumption, there is an explicit construction of the minimal exhausting disk, based on methods introduced by Matignon in [16]. This is worked out in detail in the first author's thesis [2].

If G is not solvable then we argue by contradiction, and assume that the set \mathcal{D} of all exhausting disks does not have a minimum. To each disk D in \mathcal{D} we associate a point $x_D \in X^{\text{an}}$ on the Berkovich analytic space associated to X (essentially, x_D corresponds to the maximum norm on D). A compactness argument shows that the sequence x_D converges to a point $x_0 \in X^{\text{an}}$. Points on X^{an} fall into four different classes, see [3], §1.4. We then show that in each of these four cases we can derive a contradiction, thus proving the claim. A crucial fact used in these arguments is that ‘inertia groups are solvable’.

1.4 The argument sketched above is really quite different from the traditional proofs and requires less heavy machinery than most of them. For instance,

¹The second named author has learned the idea of reducing the semistable reduction theorem to the above statement from a lecture of Raynaud at a conference in Rennes in 2009 ([17])

we do not use étale cohomology nor the Picard functor nor resolution of singularities. Our use of rigid analytic geometry is very limited and could be easily replaced by more elementary arguments. Ultimately, our proof relies on valuation theoretic arguments. In this sense it may be considered to be similar in nature to Temkin's proof of the stable modification theorem for families of curves ([18]), although this is a much deeper and more difficult result.

The solvable case of our proof is truly constructive and gives a concrete and useful algorithm to compute semistable reduction of curves in the cases where it applies. Examples where the curve is a cyclic cover of the projective line of order p are worked out in [2]. In the nonsolvable case our argument is, as it is written down here, fundamentally nonconstructive. Nevertheless we believe that a future variant will yield a constructive and practical method as well.

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2 Semistable reduction for covers

2.1 Let K be a field which is complete with respect to a discrete valuation $v : K^\times \rightarrow \mathbb{R}$. We let R denote the valuation ring of v , $\mathfrak{m} \triangleleft R$ the maximal ideal of R and $k := R/\mathfrak{m}$ the residue field. We also let π denote a uniformizer of R ; the particular choice of π will play no role.

We make the additional assumption that the residue field k is algebraically closed. By [15], Lemma 10.4.5 this is no restriction of generality, as far as the Semistable Reduction Theorem is concerned.

We remark that our base ring R is a complete discrete valuation ring and is therefore excellent (see [15], §8.2). As a consequence, all schemes and formal schemes occurring in this paper will be automatically excellent. This fact will be used in several places throughout the paper. For instance, if A is a localization of an R -algebra of finite type, then A is also excellent.

2.2 Reduced special fiber and permanence Let X be a smooth projective and absolutely irreducible curve over K . A *pre-model* of X is a flat and proper R -scheme X_R such that $X_R \otimes_R K = X$. Note that a pre-model is a model if and only if it is normal.

Lemma 2.1 *Let X_R be a pre-model of X . Let $X_s := X_R \otimes_R k$ be the special fiber of X_R . Then X_R is normal (i.e. a model) if and only if the following two conditions hold.*

- (a) *The special fiber X_s has no embedded points (see [15], Definition 7.1.6).*
- (b) *The local ring $\mathcal{O}_{X_R, \eta}$ is a discrete valuation ring, for every generic point η of X_s .*

Proof: This follows from Serre's criterion for normality, see [15], Theorem 2.23. Indeed, (b) is equivalent to the Condition (R1), whereas (a) means that X_s satisfies Condition (S1). By *loc.cit.*, Proposition 2.11, the latter is equivalent to Condition (S2) for X_R . \square

Corollary 2.2 *Let X_R be a pre-model of X with special fiber X_s .*

- (i) *If X_s is reduced then X_R is normal.*
- (ii) *Assume that X_R is normal. Then X_s is reduced if and only if X_s is reduced in codimension zero (i.e. for every generic point $\eta \in X_s$ the local ring $\mathcal{O}_{X_s, \eta}$ is a field).*

Proof: Assume that X_s is reduced. We have to show that Condition (a) and (b) of the lemma hold true. This is obvious for Condition (a). In order to verify Condition (b), let $\eta \in X_s$ be a generic point. Then $A := \mathcal{O}_{X_R, \eta}$ is a noetherian local ring of dimension one. We have

$$\mathcal{O}_{X_s, \eta} = A/\pi A.$$

Therefore, our assumption that X_s is reduced shows that πA is the maximal ideal of A . It follows that A is a discrete valuation ring, i.e. Condition (b) of the lemma holds as well. This proves Assertion (i) of the corollary. (See [15], Lemma 1.18, for a direct proof which does not use the Serre criterion.)

For the proof of (ii) we assume that X_R is normal. Then X_s has no embedded points (Condition (b)). Since X_s has dimension one, this means that X_s is reduced if and only if it is reduced in codimension zero. \square

Let Y_R be a model of Y , and let L/K be a finite extension. Let S denote the integral closure of R in L . The *normalized base change* of Y_R to L is defined to be the normalization $Y_S := (Y_R \otimes_R S)^\sim$ of the scheme $Y_R \otimes_R S$. Clearly, Y_S is a model of Y_L .

Proposition 2.3 *Let Y_R be a model of Y . Then there exists a finite extension L/K such that the normalized base change $Y_S := (Y_R \otimes_R S)^\sim$ of Y_R to L has a reduced special fiber. Furthermore, if L'/L is any finite extension, then the usual base change $Y_S \otimes_S S'$ is normal. (Here S and S' denote the integral closures of R in L and L' .)*

Proof: Let η be a generic points of Y_s . Since Y_R is assumed to be normal, the local ring $A_\eta := \mathcal{O}_{Y_R, \eta}$ is a discrete valuation ring dominating R , by Condition (b) of Lemma 2.1. By Corollary 2.2 (ii), the special fiber is reduced if and only if

$$\mathcal{O}_{Y_s, \eta} = A_\eta/\pi A_\eta$$

is a field. The latter condition holds if and only if π is a uniformizer of A_η .

Let L/K be a finite extension, S the valuation ring of L , π' a uniformizer of S and $Y_S := (Y_R \otimes_R S)^\sim$ the normalized base change. Let η' be a generic

point of the special fiber of Y_S lying over η . Then the local ring $A_{\eta'} := \mathcal{O}_{Y_S, \eta'}$ is a discrete valuation ring dominating S . Moreover, $A_{\eta'}$ is a direct factor of the integral closure of $A_\eta \otimes_R S$. Now it follows from a theorem of Epp ([12]) that there exists a finite extension L/K such that π' is a uniformizer for $A'_{\eta'}$, for every generic point η' of the special fiber of Y_S . This implies that the special fiber of Y_S is reduced.

Recall that the residue field k of K is assumed to be algebraically closed. So every finite extension of K has residue field k . In particular, if L'/L is a further finite extension, then Y_S and $Y_S \otimes_S S'$ have the same special fiber, which is reduced. Now it follows from Corollary 2.2 (i) that $Y_S \otimes_S S'$ is normal. This completes the proof of the proposition. \square

Definition 2.4 A model X_R of X with reduced special fiber is called *permanent*.²

Proposition 2.3 says that every given model of X becomes permanent after normalized base change to a suitable finite extension of the base field. Furthermore, permanent models are permanent in the sense that their special fibers are unchanged under any finite extension of the base field. Therefore, we may always assume, while proving the Semistable Reduction Theorem, that any given model is permanent.

2.3 Let Y_R be a permanent model of Y with special fiber Y_s . Let \tilde{Y}_s denote the normalization of Y_s . Note that \tilde{Y}_s is a smooth (not necessarily connected) k -curve and that we have a finite morphism $p : \tilde{Y}_s \rightarrow Y_s$ which is an isomorphism when restricted to the smooth part of Y_s . For a closed point $y \in Y_s$ we set

$$\delta_y := \dim_k(p_* \mathcal{O}_{\tilde{Y}_s} / \mathcal{O}_{Y_s})_y$$

and

$$m_y := |p^{-1}(y)|.$$

It is easy to see that $\delta_y \geq m_y - 1$.

Proposition 2.5 Let y be closed point of Y_s .

- (i) The point y is a smooth point of Y_s if and only if $\delta_y = 0$.
- (ii) The point y is an ordinary double point of Y_s if and only if $\delta_y = 1$ and $m_y = 2$.
- (iii) We have

$$g = 1 + \sum_V (g_V - 1) + \sum_{y \in Y_s} \delta_y. \quad (1)$$

Here V runs over the irreducible components of \tilde{Y}_s and g_V denotes the genus of the normalization of V .

²This terminology is also inspired by Raynaud's talk [17].

Proof: This is well known. See for e.g. [15], Proposition 7.5.4 and Proposition 7.5.15. \square

2.4 Let Y_R be a permanent model of Y . Let Y_s denote the special fiber of Y_R .

Definition 2.6 A *modification* of Y_R is an R -morphism $f : Y'_R \rightarrow Y_R$, where Y'_R is another model of Y and f is the identity on the general fiber. The modification f is called *permanent* if Y'_R is permanent. The subset of Y_s where f is not an isomorphism is called the *center* of f .

Note that a modification $f : Y'_R \rightarrow Y_R$ has connected fibers because Y_R is normal. An irreducible component W of the special fiber Y'_s of Y'_R is called *exceptional* if $f(W)$ is a closed point of Y_s . The union of the exceptional components is called the *exceptional divisor*. The union of the irreducible components of Y'_s which are not exceptional is called the *strict transform* of Y_s . The normalization $p : \tilde{Y}_s \rightarrow Y_s$ factors through a finite map $p' : \tilde{Y}_s \rightarrow Y'_s$. The image of p' is precisely the strict transform.

Definition 2.7 A permanent modification $f : Y'_R \rightarrow Y_R$ is called *simple* if the following holds:

- (i) Every exceptional component intersects the strict transform.
- (ii) Every point of intersection of an exceptional component with the strict transform is an ordinary double point of Y'_s .

Definition 2.8 Let $f : Y'_R \rightarrow Y_R$ be a simple modification and $y \in Y_s$ a singular point. Then f is called an *improvement at y* if for every closed point $y' \in f^{-1}(y)$ which does not lie on the strict transform of Y_s we have

$$\delta_{y'} < \delta_y \quad \text{or} \quad \delta_{y'} = \delta_y, \quad m_{y'} > m_y.$$

Lemma 2.9 Let $f : Y'_R \rightarrow Y_R$ be a simple modification and $y \in Y_s$ a singular point which lies in the center of f . Assume that f is not an improvement at y . Then:

- (i) The fiber $W := f^{-1}(y)$ has a unique singular point y' .
- (ii) Every irreducible component of W intersects the strict transform of Y_s in a unique point distinct from y' .
- (iii) The normalization of every irreducible component of W has genus zero and contains a unique point lying over y' .

Proof: Let $Z \subset Y'_s$ denote the strict transform of Y_s . Write $W \cap Z = \{y'_1, \dots, y'_m\}$. By Condition (iii) of Definition 2.7, each point y'_i is an ordinary double point of Y'_s and hence a smooth point of Z and of W . It follows that the finite map $p' : \tilde{Y}_s \rightarrow Z$ induces a bijection between the fiber $p'^{-1}(y)$ (where $p : \tilde{Y}_s \rightarrow Y_s$ is the normalization) and the set $W \cap Z$. Hence $m = m_y$, in the notation of §2.3. We also have $\delta_{y'_i} = 1$ by Proposition 2.5 (ii).

Set $U := W - Z$ and let S denote the set of irreducible components of W . Comparing the two expressions for the genus g obtained by applying Proposition 2.5 (iii) to Y_R and to Y'_R , one easily shows that

$$\delta_y = m_y - |S| + \sum_{V \in S} g_V + \sum_{y' \in U} \delta_{y'}. \quad (2)$$

It follows from Condition (ii) of Definition 2.7 that every component $V \in S$ contains at least one of the points y'_i (and this y'_i lies on no other component in S). Therefore, $|S| \leq m_y$, and so (2) gives the inequality

$$\delta_y \geq \sum_{V \in S} g_V + \sum_{y' \in U} \delta_{y'}. \quad (3)$$

Since by assumption f is not an improvement at y , there exists at least one point $y' \in U$ with $\delta_{y'} \geq \delta_y$. But then (3) implies that $\delta_{y'} = \delta_y \geq 1$, $\delta_{y''} = 0$ for all $y'' \in U \setminus \{y'\}$ and $g_V = 0$ for all $V \in S$. This proves (i) and the first half of (iii). Our argument also shows that the inequality (3) is actually an equality. It follows that $m_y = |S|$, and this proves (ii). Since W is connected and y' the only singular point, every irreducible component must pass through y' . This shows that $|S| \leq m_{y'}$. Finally, our assumption that f is not an improvement implies $m_{y'} \leq m_y = |S|$, which proves the second half of (iii). \square

2.5 Let $K(Y)$ denote the function field of Y . The extension $K(Y)/K$ is a regular extension of transcendence degree one. Therefore, there exists an element $x \in K(Y)$ such that $K(Y)/K(x)$ is finite and separable. The choice of x corresponds to a finite separable morphism $\phi : Y \rightarrow X := \mathbb{P}_K^1$ (we identify the rational function field $K(x)$ with the function field of \mathbb{P}_K^1).

We will prove the following ‘relative version’ of the Semistable Reduction Theorem.

Theorem 2.10 *Let $\phi : Y \rightarrow X := \mathbb{P}_K^1$ be as above. Then (after replacing K by a finite extension) there exists a semistable model X_R of X such that the normalization Y_R of X_R in $K(Y)$ is a semistable model of Y .*

Obviously, Theorem 2.10 implies Theorem 1.1. The proof of Theorem 2.10 will occupy the rest of this paper. We start with a preliminary remark.

Proposition 2.11 *For the proof of Theorem 2.10 we may assume that the extension $K(Y)/K(x)$ is Galois.*

Proof: Let \tilde{Y} be the smooth projective curve whose function field $K(\tilde{Y})$ is the Galois closure of $K(Y)/K(x)$. Let $G := \text{Gal}(K(\tilde{Y})/K(x))$ denote the Galois group and $H \subset G$ the subgroup corresponding to $K(Y)$. Then G acts on \tilde{Y} and the natural map $\tilde{Y} \rightarrow Y$ identifies Y with the quotient curve \tilde{Y}/H .

Let X_R be a semistable model of X such that its normalization \tilde{Y}_R in $K(\tilde{Y})$ is a semistable model of \tilde{Y} . Then $Y_R := \tilde{Y}_R/H$ is a semistable model of Y , see [15], Proposition 10.3.48. Moreover, the map $\phi : Y \rightarrow X$ extends to a finite map $Y_R \rightarrow X_R$. Since Y_R is normal, Y_R is the normalization of X_R in $K(Y)$. This proves the proposition. \square

2.6 We can now formulate our strategy to construct a semistable model of Y . We choose a finite separable map $\phi : Y \rightarrow X := \mathbb{P}_K^1$. By Proposition 2.11, we may assume that ϕ is a Galois cover, with Galois group G . Let X_R be a semistable model of X , and let Y_R denote the normalization of X_R in Y . Then Y_R is a G -equivariant model of Y such that $Y_R/G = X_R$. By Proposition 2.3 we may assume that X_R and Y_R are permanent. Let $\phi_s : Y_s \rightarrow X_s$ be the finite map induced by ϕ . Note that ϕ_s is invariant under the induced action of G on Y_s and that the induced map $Y_s/G \rightarrow X_s$ is a homeomorphism. However, it may not be an isomorphism. It is an isomorphism only if G acts faithfully on each irreducible component of Y_s .

Definition 2.12 A closed point $x \in X_s$ is called *critical* with respect to ϕ if the inverse image $\phi_s^{-1}(x)$ contains a non-nodal point of Y_s . We say that the semistable model X_R is *admissible* with respect to ϕ if every critical point x is a smooth point of X_s .

The following proposition is the crucial step in our proof of the Semistable Reduction Theorem.

Proposition 2.13 *Let X_R be an admissible semistable model of X , relative to ϕ . Let Y_R be the normalization of X_R in Y (which we assume is permanent). Let $x \in X_s$ be a critical point. Then (after replacing K by a finite extension) there exists a simple modification $f : X'_R \rightarrow X_R$ with center x such that the following holds.*

- (i) *Let Y'_R denote the normalization of X'_R in Y (which we assume is permanent). Then the induced map $g : Y'_R \rightarrow Y_R$ is a simple modification.*
- (ii) *The modification g is an improvement at every point $y \in \phi_s^{-1}(x)$.*

The proof of this proposition is given in the remaining sections, starting with §3.

2.7 Assuming Proposition 2.13 for the moment we can give a proof of Theorem 2.10. Let $\phi : Y \rightarrow X = \mathbb{P}_K^1$ be as above. Then the smooth model $X_R := \mathbb{P}_R^1$ is clearly admissible with respect to ϕ . Let Y_R be the normalization of X_R in Y .

If Y_R is a semistable model then we are done. Otherwise, there exists a critical point $x \in X_s$. Since the inverse image $\phi_s^{-1}(x)$ is a single G -orbit, the invariant δ_y defined in §2.3 is the same for all $y \in \phi_s^{-1}(x)$. Hence we may write $\delta_x := \delta_y$.

Let $f : X'_R \rightarrow X_R$ be a simple modification as in Proposition 2.13, relative to x . Since x is a smooth point of X_s , the fiber $f^{-1}(x) \subset X'_s$ is a smooth curve of genus zero, intersecting the strict transform of X_s transversally in a unique point x'_0 (this follows easily from (2)). In particular, the model X'_R is semistable.

If the normalization Y'_R of X'_R in Y is semistable, then we are done. Otherwise, let $x' \in f^{-1}(x)$ be a critical point with respect to ϕ . It follows from Condition (ii) in Proposition 2.13 and Definition 2.7 that x' is a smooth point of X_s (this shows that the model X'_R is admissible for ϕ). By Definition 2.8 we have $\delta_{x'} < \delta_x$ or $\delta_{x'} = \delta_x$, $m_{x'} > m_x$.

All in all we see that by repeated application of Proposition 2.13 we can either strictly decrease the invariant δ_x or keep it constant and increase m_x . Since $\delta_x \geq 0$ and $m_x \leq \delta_x + 1$, this process has to stop after a finite number of steps. It ends with a semistable model X_R whose normalization in Y is a semistable model of Y . \square

3 The rigid analytic point of view

3.1 We keep the assumption on our base field K . In the context of rigid analytic geometry it is more convenient to work with an absolute value instead of with an (exponential) valuation. We therefore choose a real constant $0 < q < 1$ and set $|a| := q^{v(a)}$ for $a \in K$.

Let X be a smooth projective K -curve. We let X^{rig} denote the rigid analytic space associated to X , see e.g. [6] or [13]. Recall that the set underlying X^{rig} is simply the set of closed points of X .

Let X_R be a permanent model of X . Given a point $x \in X^{\text{rig}}$, its scheme theoretic closure in X_R intersects the special fiber X_s in a unique point $\bar{x} \in X_s$, called the *specialization* of x . The resulting map

$$\text{sp}_{X_R} : X^{\text{rig}} \rightarrow X_s$$

is surjective and is called the *specialization map* of the model X_R .

Let $Z \subset X_s$ be a locally closed subscheme. Then the inverse image

$$]Z[_{X_R} := \text{sp}_{X_R}^{-1}(Z) \subset X^{\text{rig}}$$

is an open set in the G -topology for X^{rig} and hence is a smooth rigid analytic K -space. We call $]Z[_{X_R}$ the *tube* of Z in X_R . See e.g. [5], §1.

Remark 3.1 Let $Z \subset X_s$ be a locally closed subscheme. Let $\mathcal{X} := X_R|_{\widehat{Z}}$ be the formal completion of X_R along Z .

- (i) The tube $]Z[_{X_R}$ is canonically isomorphic to the *generic fiber* \mathcal{X}_K of \mathcal{X} as constructed in [5], §1 (see also [10], §7).
- (ii) Let $\mathcal{O}_{X^{\text{rig}}}^\circ$ denote the subsheaf of the structure sheaf on X^{rig} consisting of functions that are bounded by 1. Then we have a canonical isomorphism

$$\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \xrightarrow{\sim} \Gamma(]Z[_{X_R}, \mathcal{O}_{X^{\text{rig}}}^\circ),$$

see [10], Theorem 7.4.1.

- (iii) The definition of $]Z[_{X_R}$ is compatible with base change to any finite extension L/K . More precisely, we have a canonical isomorphism

$$]Z[_{X_R \otimes_R L} \cong]Z[_{X_R \otimes_R S}.$$

Here S denotes the integral closure of R in L , and we identify the special fiber of X_R with that of $X_R \otimes_R S$. Note that $X_R \otimes_R S$ is again a permanent model by Proposition 2.3.

- (iv) The tube $]Z[_{X_R}$ is connected if and only if Z is connected. This follows immediately from (ii): any idempotent function on $]Z[_{X_R}$ is analytic and bounded by 1 and hence gives rise to an idempotent function on $Z = \mathcal{X}^{\text{red}}$.
- (v) Combining (iii) and (iv) shows that the connected components of $]Z[_{X_R}$ are absolutely connected.
- (vi) Suppose that $Z = \text{Spec}(\bar{A})$ is an affine open subset of X_s . Then $\mathcal{X} = \text{Spf}(A)$ is an affine formal scheme, where A is flat and topologically of finite presentation over R , and is complete with respect to the π -adic topology (so A is *admissible* in the terminology of [8]). Therefore, $A_K := A \otimes_R K$ is an affinoid K -algebra. Now it follows from the construction that $]Z[_{X_R} = \mathcal{X}_K = \text{Spm}(A_K)$ is an affinoid subdomain of X^{rig} .

In this special case, (ii) says that

$$A = A_K^\circ := \{f \in A_K \mid \|f\| \leq 1\}.$$

Here $\|\cdot\|$ denotes the maximum norm on the affinoid $\text{Spm}(A_K)$. It follows that $\bar{A} = A/\pi A$ and that $Z = \text{Spec}(\bar{A})$ is the *canonical reduction* of the affinoid domain $]Z[_{X_R}$ (in the sense of [13], §4.8).

3.2 Let us now consider the case where $Z = \{x\}$ consists of a single closed point of X_s . Then the tube $\mathbf{X} :=]x[_{X_R}$ is called the *residue class* of x (with respect to the model X_R). Let $A := \hat{\mathcal{O}}_{X_R, x}$ denote the complete local ring of the model X_R at x . By Remark 3.1, \mathbf{X} can be identified with the generic fiber of the formal R -scheme $\mathcal{X} = \text{Spf}(A)$. Moreover, A can be identified with the ring of analytic functions on \mathbf{X} bounded by 1. It follows that the residue class \mathbf{X} depends, as a rigid analytic space, only on the completion of X_R at x .

Definition 3.2 An open analytic curve over K is a rigid analytic K -space X which becomes isomorphic, after a finite extension of K , to a residue class $]x[_{X_R}$, where X_R is a permanent model of a smooth projective K -curve X and $x \in X_s$ is a closed point of the special fiber. The formal R -scheme $\mathcal{X} = \mathrm{Spf}(A)$, where $A = \Gamma(X, \mathcal{O}_X^\circ)$, is called the *canonical formal model* of X . A *boundary point* of X is a generic point of $\mathrm{Spec}(A/\pi A)$. The set of boundary points of X is denoted by ∂X .

A boundary point $\eta \in \partial X$ gives rise to a discrete valuation on $\mathrm{Frac}(A)$. Its residue field $k(\eta)$ is a complete discrete valuation field containing k . It is thus isomorphic to $k((t))$.

Suppose that $X =]x[_{X_R}$ is a residue class as above. Then $A = \hat{\mathcal{O}}_{X_R, x}$. It follows that a boundary point $\eta \in \partial X$ corresponds to a local branch of X_s through x . We obtain a natural bijection between ∂X and the fiber $p^{-1}(x)$, where $p : \tilde{X}_s \rightarrow X_s$ is the normalization of X_s . We write $\bar{\eta} \in \tilde{X}_s$ for the point corresponding to $\eta \in \partial X$.

Definition 3.3 An open analytic curve X over K is called an *open disk* if it is isomorphic to the standard open unit disk, i.e. to the rigid K -space

$$\{t \in \mathbb{A}_K^1 \mid |t| < 1\}.$$

It is called an *open annulus* if it is isomorphic to

$$\{u \in \mathbb{A}_K^1 \mid \epsilon < u < 1\},$$

for some $\epsilon \in \sqrt{|K^\times|}$, with $\epsilon < 1$.

Proposition 3.4 Let X_R be a permanent model of a smooth projective curve over K . Let $x \in X_s$ be a closed point of the special fiber and let $X :=]x[_{X_R}$ denote the residue class of x . Then x is a smooth point (resp. an ordinary double point) of X_s if and only if X is an open disk (resp. an open annulus).

Proof: (compare with [7], Proposition 2.2 and 2.3) Suppose first that $X_R = \mathbb{P}_R^1$ is the projective line over R and $x := 0 \in X_s = \mathbb{P}_k^1$ the origin. It is then easy to see that the residue class of x is the standard open unit disk, and that $\hat{\mathcal{O}}_{X_R, x} = R[[t]]$, where t is the standard parameter on \mathbb{A}_R^1 .

Now let X be any smooth K -curve, X_R a permanent model and $x \in X_s$. By the above, the residue class $]x[_{X_R}$ is an open disk if and only if $\hat{\mathcal{O}}_{X_R, x} \cong R[[t]]$. But the latter holds if and only if x is a smooth point. This proves the first equivalence.

The proof of the second equivalence is similar, but we have to be more careful about the role of the base field K . As before, we can realize the standard open annulus

$$\{u \in \mathbb{A}_K^1 \mid \epsilon < u < 1\}$$

as the residue class of a point x on the special fiber of a model X_R of \mathbb{P}_K^1 . However, the model X_R is permanent if and only if $\epsilon \in |K^\times|$. If this is the case,

then

$$\hat{\mathcal{O}}_{X_{R,x}} = R[[u, v \mid uv = a]],$$

where $a \in K^\times$ is any element with $|a| = \epsilon$. With this in mind, the proof of the second equivalence is analogous to the proof of the first. \square

3.3 We need a good notion of finite (Galois) covers of open analytic curves.

Definition 3.5 Let X be an absolutely connected open analytic curve over K . An *admissible cover* of X is a finite and flat morphism $\phi : Y \rightarrow X$ of rigid-analytic K -spaces, such that Y is also an open analytic curve. The cover ϕ is called a *regular Galois cover* if Y is absolutely connected and the automorphism group $G := \text{Aut}(\phi)$ has order $\deg(\phi)$ (note that $\deg(\phi) \in \mathbb{N}$ is well defined).

Let $\phi : Y \rightarrow X$ be an admissible cover. Let $A := \Gamma(X, \mathcal{O}_X^\times)$ and $B := \Gamma(Y, \mathcal{O}_Y^\times)$. Then A is a normal and complete local domain with residue field k , and B is a finite A -algebra. Moreover, B is a normal complete semilocal ring, of the form $B = \bigoplus_i B_i$, where each B_i is a normal complete local domain with residue field k . After extending the base field K we may assume that $A/\pi A$ and $B/\pi B$ are reduced. Then ϕ is a regular Galois cover if and only if B is a domain and the field extension $\text{Frac}(B)/\text{Frac}(A)$ is Galois. If this is the case, then the Galois group of ϕ can be identified with the Galois group of $\text{Frac}(B)/\text{Frac}(A)$.

Let $\phi : Y \rightarrow X$ be a regular Galois cover, with Galois group G . Let $H \subset G$ be a subgroup. Then the quotient $Z := Y/H$ is again an absolutely irreducible open analytic curve. The induced maps $Y \rightarrow Z$ and $Z \rightarrow X$ are admissible covers. Moreover, $Y \rightarrow Z$ is a regular Galois cover with Galois group H , and $Z \rightarrow X$ is Galois if and only if H is a normal subgroup of G .

3.4 We now formulate our main result (Theorem 3.9), and show that it implies the Semistable Reduction Theorem. Let X be an open disk over K and $\phi : Y \rightarrow X$ a regular Galois cover. Let G denote the Galois group of ϕ . (The assumption that X is a disk will be slightly relaxed in §4, but it will again be in force in §5).

Definition 3.6 (i) A *parameter* for the open disk X is an element $t \in A$ such that $A = R[[t]]$.

(ii) A subset $D \subset X$ is called a *closed disk* if there exists a parameter t for the open disk X and a real number ϵ , $0 < \epsilon < 1$, such that

$$D = X(|t| \leq \epsilon).$$

If D is also an affinoid subdomain then it is called an *affinoid disk* (this is the case iff $\epsilon \in \sqrt{|K^\times|}$).

(iii) An affinoid disk $D \subset X_x$ is called *exhausting* (with respect to ϕ) if the complement $Y \setminus \phi^{-1}(D)$ is the disjoint union of open annuli.

We let \mathcal{D} denote the set of all exhausting affinoid disks $D \subset X$.

Lemma 3.7 *Let t be a parameter for the open disk X . Let $\epsilon, \epsilon' \in \sqrt{|K^\times|}$ with $0 < \epsilon < \epsilon' < 1$.*

- (i) *If $X(|t| \leq \epsilon)$ is exhausting then $X(|t| \leq \epsilon')$ is exhausting as well.*
- (ii) *There exists a constant $\epsilon_0 < 1$, such that $X(|t| \leq \epsilon)$ is exhausting, for all $\epsilon \geq \epsilon_0$.*

Proof: This follows from [7], Lemma 2.4. \square

Corollary 3.8 *The set \mathcal{D} is nonempty. Moreover, if $D' \subset X$ is an affinoid disk containing an element $D \in \mathcal{D}$, then $D' \in \mathcal{D}$.*

The following theorem is a ‘local’ version of Proposition 2.13 and is really the main result of the present paper.

Theorem 3.9 *Let $\phi : Y \rightarrow X$ be a regular Galois cover of the open disk. Assume that Y is not an open disk. Then the set \mathcal{D} of all affinoid disks $D \subset X$ which are exhausting with respect to ϕ has a unique minimal element.*

The proof is given in §4 and §5, after some preliminary remarks in §3.5. In §3.6 we show that Theorem 3.9 implies the Semistable Reduction Theorem.

3.5 We fix a regular G -Galois cover $\phi : Y \rightarrow X$ of the open disk. We let $\mathcal{X} = \mathrm{Spf}(A)$ and $\mathcal{Y} = \mathrm{Spf}(B)$ denote the canonical formal models of X and Y . We let $\eta \in \partial X$ denote the unique boundary point of X . We let $k[\eta] \subset k(\eta)$ denote the valuation ring of the residue field of η . In this section, we consider η as a morphism of formal schemes $\eta : \mathrm{Spf}(k[\eta]) \rightarrow \mathcal{X}$.

Let $D \subset X$ be an affinoid disk. It gives rise to a diagram of formal R -schemes

$$\begin{array}{ccc} \mathcal{Y}' & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{X}' & \longrightarrow & \mathcal{X}, \end{array}$$

as follows. Let $t \in A$ be a parameter and $\epsilon \in |K^\times|$ such that $D = X(|t| \leq \epsilon)$. Choose an element $a \in R$ with $v(a) = \epsilon$ and let $\mathcal{X}' \rightarrow \mathcal{X}$ be the formal blowup of the ideal $I := (t, a) \triangleleft A$. Let $Z \subset \mathcal{X}'$ be the exceptional fiber (it is equal to the reduced subscheme $(\mathcal{X}')^{\mathrm{red}}$, and it is isomorphic to \mathbb{P}_k^1). The morphism $\xi : \mathrm{Spf}(k[\eta]) \rightarrow \mathcal{X}$ lifts uniquely to a morphism $\xi' : \mathrm{Spf}(k[\eta]) \rightarrow \mathcal{X}'$. Let $z \in Z$ denote the image of ξ' and $Z^\circ := Z \setminus \{z\}$. Then

$$D =]Z^\circ[_{\mathcal{X}'}.$$

Let \mathcal{Y}' be the normalization of the formal scheme \mathcal{X}' in \mathcal{Y} (see [9], §2.1). We call \mathcal{Y}' the formal model of Y induced by D . Let $W := (\mathcal{Y}')^{\mathrm{red}}$ denote

the reduced subscheme. Note that W is a connected projective k -curve. The canonical morphism $\mathcal{Y}' \rightarrow \mathcal{X}'$ restricts to a finite map $W \rightarrow Z$. Let $\partial W \subset W$ denote the inverse image of z and $W^\circ := W \setminus \partial W$. We have

$$\phi^{-1}(\mathcal{D}) =]W^\circ[_{\mathcal{Y}'}$$

It follows that \mathcal{D} is exhausting with respect to ϕ if and only if the residue classes $]w[_{\mathcal{Y}'}$ are open annuli, for all $w \in \partial W$. Actually, since G acts transitively on the set ∂W , it suffices that this holds for one $w \in \partial W$.

Given a boundary point $\xi \in \partial \mathcal{Y}$ the morphism $\xi : \mathrm{Spf}(k[\xi]) \rightarrow \mathcal{Y}$ lifts uniquely to $\xi' : \mathrm{Spf}(k[\xi]) \rightarrow \mathcal{Y}'$, and the image $\tilde{\xi} \in W$ of ξ' lies in ∂W . We obtain a surjective G -equivariant map

$$\partial \mathcal{Y} \rightarrow \partial W. \quad (4)$$

If \mathcal{D} is exhausting, then this map is a bijection.

Lemma 3.10 *Assume that $\mathcal{D} \in \mathcal{D}$ is exhausting. Then the following holds.*

- (i) *The affinoid $\mathcal{E} := \phi^{-1}(\mathcal{D})$ is absolutely connected.*
- (ii) *The cover \mathcal{Y} is an open disk if and only if \mathcal{E} is a closed disk.*
- (iii) *Assume that \mathcal{Y} is not an open disk. Then the set \mathcal{D} of all exhausting closed disks is totally ordered with respect to inclusion.*

Proof: If \mathcal{D} is exhausting, then $W^\circ \subset W$ is the complement of a finite set of smooth points of W . Since W is connected it follows that W° is connected as well. By Remark 3.1 (iv),(v) this implies that $\phi^{-1}(\mathcal{D}) =]W^\circ[_{\mathcal{Y}'}$ is absolutely connected, proving (i).

The affinoid $\phi^{-1}(\mathcal{D}) =]W^\circ[_{\mathcal{Y}'}$ is a closed disk if and only if $W^\circ \cong \mathbb{A}_k^1$. Under the assumption that \mathcal{D} is exhausting, this holds if and only if $W \cong \mathbb{P}_k^1$, \mathcal{Y} has a unique boundary point ξ and the intersection of W with the corresponding formal subscheme $\mathrm{Spf} k[\xi] \subset \mathcal{Y}'$ is transversal. By Castelnuovo's criterion this holds if and only if $\mathcal{Y} \cong \mathrm{Spf} R[[s]]$, i.e. \mathcal{Y} is an open disk. This proves (ii).

To prove (iii) we let $\mathcal{D}' \subset \mathcal{D}$ be another exhausting disk, disjoint from \mathcal{D} . We then have to show that \mathcal{Y} is an open disk. We may write $\mathcal{D} = \{|t| \leq \epsilon\}$, for some parameter t and some $\epsilon \in |K^\times|$, $0 < \epsilon < 1$. Then there exists an $\epsilon' > \epsilon$ such that \mathcal{D}' is contained in a residue class of the affinoid $\mathcal{A} := \{|t| = \epsilon'\}$ (a closed annulus of thickness 0). Let $\mathcal{X}' \subset \mathcal{A}$ denote the residue class containing \mathcal{D}' . The inverse image $\mathcal{Y}' := \phi^{-1}(\mathcal{X}')$ is the disjoint union of residue classes of the affinoid $\phi^{-1}(\mathcal{A})$. Since \mathcal{D} is exhausting, $\phi^{-1}(\mathcal{A})$ is a disjoint union of closed annuli of thickness 0 (this follows from [7], Lemma 2.4). In particular, $\phi^{-1}(\mathcal{A})$ is an affinoid with good reduction, and hence \mathcal{Y}' is the disjoint union of open disks. Applying (ii) to the connected components of the cover $\mathcal{Y}' \rightarrow \mathcal{X}'$ we see that $\phi^{-1}(\mathcal{D}')$ is a disjoint union of closed disks. By (i), $\phi^{-1}(\mathcal{D}')$ is connected and hence an open disk. Applying (ii) again shows that \mathcal{Y} is an open disk, as desired. This finishes the proof of the lemma. \square

3.6 For the rest of this section we will show that Theorem 3.9 implies Proposition 2.13 and hence the Semistable Reduction Theorem (as explained in §2.7).

We return to the situation considered in §2.5. Let Y be a smooth projective and absolutely irreducible curve over K and let $\phi : Y \rightarrow X := \mathbb{P}_K^1$ be a finite separable and nonconstant morphism to the projective line X . We assume that ϕ is a Galois cover, and let G denote its Galois group. The cover $\phi : Y \rightarrow X$ induces a finite morphism of rigid analytic K -spaces $\phi^{\text{rig}} : Y^{\text{rig}} \rightarrow X^{\text{rig}}$.

Let X_R be a semistable model of X which is admissible for ϕ (Definition 2.12). Let Y_R denote the normalization of X_R in Y . We assume that Y_R is a permanent model of Y . Let $x \in X_s$ be a critical point. By assumption $x \in X_s$ is a smooth point. It follows that the residue class $\mathbf{X} :=]x[_{X_R}$ is an open disk (Proposition 3.4). Set $A := \hat{\mathcal{O}}_{X_R, x}$.

Let $\mathbf{Y} := \phi^{-1}(\mathbf{X}) \subset Y^{\text{rig}}$ denote the inverse image of the residue class \mathbf{X} . This is an open rigid subspace of Y^{rig} which is invariant under the action of the Galois group G . In fact,

$$\mathbf{Y} = \bigcup_{y \in \phi_s^{-1}(x)} \mathbf{Y}_y, \quad \mathbf{Y}_y :=]y[_{Y_R}$$

is the disjoint union of the residue classes of the points $y \in Y_s$ lying over x . The assumption that x is a critical point is equivalent to the statement that the residue classes \mathbf{Y}_y are not isomorphic to open disks. Therefore, for any y the induced cover $\phi_y : \mathbf{Y}_y \rightarrow \mathbf{X}$ is a Galois covers with Galois group $G_y = \text{Stab}_G(y)$ which satisfies the hypotheses of Theorem 3.9.

Let $\mathbf{D} \subset \mathbf{X}$ be an affinoid disk. After replacing K by a finite extension we may assume that \mathbf{D} contains a K -rational point P . Choose an element $t \in \mathcal{O}_{X_R, x}$ which has a simple zero at P and no other zero on the residue class \mathbf{X} . Then t is a parameter for the open disk \mathbf{X} . Moreover, $\mathbf{D} = \mathbf{X}(|t| \leq \epsilon)$ for some $\epsilon \in \sqrt{|K^\times|}$ with $0 < \epsilon < 1$. After a further extension of K we may assume that $\epsilon \in |K^\times|$.

Choose an element $a \in R$ with $|a| = \epsilon$, and let $f : X'_R \rightarrow X_R$ be the blowup with center x of the ideal $(t, a) \triangleleft \mathcal{O}_{X_R, x}$. Let $Z := f^{-1}(x)$ be the exceptional divisor and $\mathcal{X}' := X'_R|_{\widehat{Z}}$ the formal completion of X'_R along Z . The natural morphism of formal R -schemes $\mathcal{X}' \rightarrow \mathcal{X} = \text{Spf}(A)$ is the formal blowup of the ideal $(t, a) \triangleleft A$, see [8], §2. By the explicit description of formal blowups in *loc.cit.* one sees that $Z \cong \mathbb{P}_k^1$ is a smooth curve of genus zero which intersects the strict transform of Y_s in a unique point z and that z is an ordinary double point of Y'_s . In particular, $f : X'_R \rightarrow X_R$ is a simple modification. Furthermore,

$$\mathbf{D} =]Z^\circ[_{X'_R}, \quad \text{with } Z^\circ := Z - \{z\}.$$

Let $g : Y'_R \rightarrow Y_R$ denote the modification induced by f (i.e. Y'_R is the normalization of X'_R in Y). After a finite extension of the base field K we may assume that the model Y'_R is permanent. Let $W \subset Y'_s$ denote the exceptional divisor of g . Since $Y'_R \rightarrow X'_R$ is finite, it restricts to a finite map $W \rightarrow Z$. Moreover, the set $\partial W \subset W$ of points where W intersects the strict transform of Y'_s is precisely the inverse image of z in W . It follows that

$$\phi^{-1}(\mathbf{D}) =]W^\circ[_{Y'_R}, \quad \text{with } W^\circ := W \setminus \partial W.$$

We say that g is the modification of Y_R induced by D .

Note that we have a natural surjective map

$$\partial Y \rightarrow \partial W \quad (5)$$

mapping a boundary point $\xi \in \partial Y$ first to a point $\bar{\xi} \in \tilde{Y}_s$ on the normalization of Y_s (see §3.2) and then to its image under the map $p' : \tilde{Y}_s \rightarrow Y'_s$ discussed after Definition 2.6. If the modification g is simple, then this map is a bijection.

We call the affinoid disk D *exhausting* if it is exhausting with respect to the cover $\phi_y : Y_y \rightarrow X$, in the sense of Definition 3.6 and for some $y \in \phi^{-1}(x)$ (in fact this condition is independent of y). As in §3.4 we let \mathcal{D} denote the set of all exhausting affinoid disks $D \subset X$.

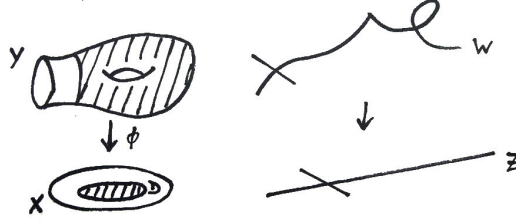


Figure 1: Here D is minimal exhausting.

Proposition 2.13 is now an immediate consequence of Theorem 3.9 and the following lemma.

Lemma 3.11 *Let $D \subset X$ be an affinoid disk and $g : Y'_R \rightarrow Y_R$ the induced modification.*

- (i) *The modification g is simple if and only if D is exhausting.*
- (ii) *Assume D is exhausting. Then $g : Y'_R \rightarrow Y_R$ is an improvement at every point $y \in \phi^{-1}(x)$ if and only if D is a minimal element of \mathcal{D} .*

Proof: Every irreducible component of W is a finite cover of Z and therefore contains a point $w \in \partial W$ lying above z . As remarked above, these are precisely the points where W intersects the strict transform of Y_s . It follows that g satisfies Condition (i) of Definition 2.7. Condition (ii) holds if and only if every point $w \in \partial W$ is an ordinary double point of Y'_s . But the residue classes $]w[_{Y'_R}$ are precisely the connected components of $Y \setminus \phi^{-1}(D)$. Therefore, Condition (ii) of Definition 2.7 holds if and only if D is exhausting (here we use Proposition 3.4). This proves (i).

For the proof of (ii) we assume that g is not an improvement at some $y \in \phi^{-1}(x)$. Set $W_y := g^{-1}(y)$; this is a connected component of W . Set $W_y^\circ := W_y \cap W^\circ$. By Lemma 2.9, W_y has a unique singular point y' . All irreducible components of W_y have geometric genus zero and are smooth outside

y' . Moreover, they intersect the strict transform in a unique point (which is an ordinary double point distinct from y') and have a unique branch passing through y' .

Let $x' \in Z^\circ$ denote the image of y' . Then x' is the only critical point with respect to ϕ which lies on Z . Let $X' :=]x'[_{X'_R}$ denote the residue class of x' and let $Y' := \phi^{-1}(X') =]\phi^{-1}(x')[_{Y'_R}$ be the inverse image. Applying Lemma 3.7 to the restriction $\phi' := \phi|_{Y'} : Y' \rightarrow X'$ we find an affinoid disk $D' \subset X'$ which is exhausting with respect to ϕ' . We claim that D' , as an affinoid disk in X , is exhausting with respect to ϕ . Since D' is strictly contained in D , this claim would prove the ‘if’ part of (ii).

To prove the claim, we consider the simple modification $g' : Y''_R \rightarrow Y'_R$ induced from D (as an affinoid disk in X'). The center of g' is precisely the singular locus of W . Let $W' \subset Y''_s$ denote the strict transform of W . By construction,

$$X \setminus \phi^{-1}(D') =]W'[_{Y''_R}.$$

From the above description of W and the fact that g' is a simple modification we see that W' is isomorphic to the normalization of W . More precisely, W is a disjoint union of projective lines, each of which intersects the rest of Y''_s in exactly two points, which are ordinary double points Y''_s . It follows that the tube $]W'[_{Y''_R}$ is the disjoint union of open annuli. This proves the claim and hence the ‘if’-part of (ii). The ‘only if’ part is left to the reader (it is not used in the rest of the paper). \square

4 The solvable case

In this section we prove the existence of a minimal exhausting disk (Theorem 3.9) under the assumption that the Galois group G of the cover $\phi : Y \rightarrow X$ is solvable. The proof is by induction on the order of G . The base case of the induction is when G is cyclic of prime order, and this case is treated in greater detail in [2]. To make the induction step work we actually have to consider a slightly more general situation. Namely, we allow X to be either a disk or an annulus. It is then natural to replace the notion of *exhausting affinoid disk* by *separating boundary domain*.

4.1 We keep all our assumptions on the base field K . Let X be an open analytic curve over K which is either an open disk or an open annulus. Let $A = \Gamma(X, \mathcal{O}_X^\times)$ be the ring of analytic functions on X bounded by 1. Then X is the generic fiber of the formal R -scheme $\mathcal{X} = \mathrm{Spf}(A)$, see §3.1-§3.2.

Let us choose a boundary point $\eta \in \partial X$. If X is an annulus, this amounts to choosing an ‘orientation’ of X (if X is a disk, there is no choice). A *parameter* for X (with respect to η) is an element $t \in A$ which yields an isomorphism

$$X \xrightarrow{\sim} \{t \in \mathbb{A}_K^1 \mid \epsilon_0 < |t| < 1\},$$

for some $\epsilon_0 < 0$ if X is a disk and with $\epsilon_0 > 0$ if X is an annulus. If X is a disk, then this implies $A = R[[t]]$, and the new terminology agrees with the old one. If X is an annulus, then there exists an element $a \in \mathfrak{m}_R$ such that $s := a/t \in A$ is a parameter for X with respect to the boundary point distinct from η , and we have $A = R[[t, s \mid ts = a]]$.

By a *boundary domain* of X (containing η) we mean an open rigid subspace $U \subset X$ of the form

$$U = X(|t| > \epsilon),$$

where t is a parameter for X and $\epsilon \in \sqrt{|K^\times|}$, $\epsilon < 1$. So if X is an open disk, then $U = X - D$, where $D \subset X$ is an affinoid disk. In any case, U is an open annulus.

Let $\phi : Y \rightarrow X$ be a regular Galois cover of X , with Galois group G . A boundary domain $U \subset X$ is called *separating* (with respect to ϕ) if the inverse image $\phi^{-1}(U)$ is the disjoint union of open annuli. Let \mathcal{U} denote the set of all separating boundary domains. We consider \mathcal{U} as a partially ordered set by inclusion.

If X is an open disk, then a boundary domain $U \subset X$ is separating with respect to ϕ if and only if the affinoid disk $D := X \setminus U$ is exhausting with respect to ϕ . It follows that the set \mathcal{U} has a unique maximum if and only if the set \mathcal{D} has a unique minimum. Therefore, the following proposition implies Theorem 3.9 in case the Galois group G is solvable.

Proposition 4.1 *Assume that*

- (a) *The open analytic curve Y is not an open disk.*
- (b) *The Galois group G of ϕ is solvable.*

Then the set \mathcal{U} has a unique maximal element.

For the proof of Proposition 4.1 we will use induction on the order of G . The case where $G = \{1\}$ is trivial. Indeed, if $G = \{1\}$ then $X = Y$ is not a disk. It follows that X is an annulus, and that $U := X$ is the unique maximal element of \mathcal{U} . After a preliminary argument in §4.2, we prove the prime order case in §4.3–§4.4. Finally, the induction step is done in §4.5.

4.2 Let $x_1, \dots, x_r \in X$ be the pairwise distinct branch points of ϕ (which we assume to be K -rational). We claim that, in order to prove Proposition 4.1, we may assume that $r \leq 1$ if X is a disk and $r = 0$ if X is an annulus.

To prove this claim we assume that either $r \geq 2$ and X is a disk or that $r \geq 1$ and X is an annulus. Under this condition there exists a maximal boundary domain $U_0 \subset X$ containing none of the branch points. Assume, moreover, that U_0 is not separating with respect to ϕ . Then $\phi^{-1}(U_0)$ is not a union of open disks. Furthermore, a boundary domain $U \subset U_0$ is separating with respect to $\phi : Y \rightarrow X$ if and only if it is separating with respect to the cover $\phi^{-1}(U_0) \rightarrow U_0$. But $\phi^{-1}(U_0) \rightarrow U_0$ is étale by choice of U_0 . So in this case it suffices to prove the proposition under the assumption that ϕ is étale (i.e. $r = 0$).

Now consider the case that U_0 is separating. If it is maximal with this property then we are done. Hence we may assume that U_0 is not maximal. For simplicity, we also assume that X is a disk (the other case is proved similarly). Then $D_0 := X \setminus U_0$ is a non-minimal exhausting affinoid disk. In this situation it follows from Lemma 3.10 (iii) that there exists a unique residue class $X' \subset D_0$ containing all exhausting disks strictly contained in D_0 . Furthermore, for any closed affinoid disk $D \subset X'$, D is exhausting with respect to ϕ if and only if it is exhausting with respect to the restricted cover $\phi^{-1}(X') \rightarrow X'$. But by the choice of U_0 , D_0 is the smallest closed disk containing all $r \geq 2$ branch points. It follows that the residue class $X' \subset D_0$ contains strictly less than r branch points. Our claim now follows by induction on the number r of branch points.

For the rest of the proof of Proposition 4.1 we may now assume that $r \leq 1$, and that X is a disk if $r = 1$.

4.3 Let us assume that the group G is cyclic of prime order ℓ . We have to distinguish two main cases, of which the first is divided into two subcases. We start with the assumption that the characteristic of K is prime to ℓ . (For the time being, we make no assumption on the residue characteristic of K .) Then, after replacing K by some finite extension, we may also assume that K contains an ℓ th root of unity. Moreover, the cover $\phi : Y \rightarrow X$ is given generically by a Kummer equation of the form

$$y^\ell = f,$$

where $f \in A$ is not an ℓ th power. More precisely, the ring $B := \mathcal{O}(Y)^\circ$ contains an element y which satisfies the above equation and B is the normalization of A in the field $\text{Frac}(A)[y]$. Clearly, the ring B is unchanged if we divide f by an ℓ th power in A . We may therefore assume that the order of zero of f at any point $x \in X$ is strictly less than ℓ . Under this condition the zeroes of f are precisely the branch points of ϕ .

We claim that there is no branch point, i.e. that ϕ is étale. To prove this claim we assume the converse. Then X is a disk and there is exactly one branch point, by our assumption made at the end of §4.2. We choose a parameter t for X such that the unique branch point is $t = 0$. Now the Weierstrass preparation theorem shows that f is of the form $f = c(t + a_2 t^2 + \dots) \in A = R[[t]]$, with $c \neq 0$. After a finite extension of K , c is an ℓ th power, and we may therefore assume that $c = 1$. Applying [2], Lemma 1.28, one shows that $B = A[y]$ and that Y is an open disk with parameter y . But this contradicts Assumption (a) of Proposition 4.1, proving the claim.

Let us now make the additional assumption that $\ell \neq p$, i.e. that the residue characteristic of K is prime to ℓ . Let us choose a parameter t for X . If X were a disk then $A = R[[t]]$ and, by the above claim, $f = 1 + a_1 t + a_2 t^2 + \dots$. But then Hensel's lemma shows that f is an ℓ th power in A . This contradicts our assumption that ϕ is a regular Galois cover. We conclude that X is an open annulus and hence $A = R[[t, s \mid ts = a]]$. Using again Hensel's Lemma we see that $f = ct^m u$, with $c \in R$, $m \geq 1$ and u a unit of A with constant coefficient 1. As before we may assume that $c = 1$. Dividing f by a suitable power of

t^ℓ we may also assume that $m < \ell$. In this situation [2], Lemma 1.31, shows that Y is an open annulus. This means that X itself is the maximal separating boundary domain we are looking for. Therefore, Proposition 4.1 is proved in the case $|G| = \ell \neq p$.

4.4 We continue with the notation introduced in §4.3, but we now assume that $\ell = p$ is the (positive) characteristic of the residue field k of K . (We keep the assumption that $\ell = p$ is prime to the characteristic of K , but this now amounts to saying that K has characteristic zero.) In this case, Proposition 4.1 is proved in [2] (Theorem 2.1 and Theorem 4.2). We briefly sketch the proof. For simplicity, we restrict to the case where X is an open disk. The proof in the case where X is an annulus uses the same methods, but is slightly more complicated.

As we have seen before, we may assume that the cover ϕ is given generically by an equation of the form

$$y^p = f,$$

where f is a principal unit in $A = R[[t]]$. Note that Hensel's lemma is not applicable anymore, and we cannot conclude that f is a p th power.

The first step is to compute the ring B . Let $v_0 : \text{Frac}(A) \rightarrow \mathbb{Q} \cup \{\infty\}$ denote the discrete valuation corresponding to the prime ideal $\pi A \triangleleft A$ (normalized such that $v_0(p) = 1$ which implies that $v_0|_K = v$). Since $\bar{A} := A/\pi A$ is isomorphic to a power series ring $k[[t]]$, the residue field of v_0 carries a canonical discrete valuation \bar{v}_0 (normalized such that $\bar{v}_0(t) = 1$). Let

$$v_\eta : A \rightarrow (\mathbb{Q} \times \mathbb{Z}) \cup \{\infty\}$$

denote the discrete valuation of rank two obtained as the composition of v_0 with \bar{v}_0 . Here we consider the target set as an ordered group with respect to the lexicographic ordering. More explicitly: if t is a parameter for X and $g = \sum_{i=0}^{\infty} g_i t^i \in A$ then $v_\eta(g) = (\mu, m)$, where

$$\mu = \min_i v(g_i), \quad m = \min\{i \mid v(g_i) = \mu\}.$$

Proposition 4.2 *The maximum*

$$(\mu, m) := \max\{v_\eta(f - h^p) \mid h \in A^\times\}$$

exists. Furthermore:

- (i) $0 \leq \mu < p/(p-1)$ and $\mu/p \in v(K^\times)$.
- (ii) $m > 1$ and $(m, p) = 1$.
- (iii) Choose $c \in K^\times$ such that $v(c) = \mu/p$ and $h \in A^\times$ such that $v_\eta(f - h^p) = (\mu, m)$. Then $B = A[w]$, where $w := (y - h)/c$.

Proof: Let $h \in A^\times$ be given and set $(\mu, m) := v_\eta(f - h^p)$. Then $\mu = v_0(f - h^p) \leq p/(p-1)$ (otherwise, completeness of A with respect to πA would show that $f \in A^p$). Since $\mu \in v(K^\times)$ takes values in a discrete group, we may assume that μ takes the maximal possible value. Let \tilde{v}_0 be an extension of v_0 to $\text{Frac}(B)$; it corresponds to a minimal prime ideal of $\bar{B} := B/\pi B$. Our running assumption says that \bar{B} is reduced, which means that π is a prime element for \tilde{v}_0 . A simple calculation using $\mu \leq p/(p-1)$ and the equation $y^p = f$ shows that $p\tilde{v}_0(y - h) = v_0(f - h^p) = \mu$. It follows that $\mu/p \in v(K^\times)$. Choose an element $c \in K$ with $v(c) = \mu/p$ and set $g_0 := (f - h^p)/c^p \in A$. Then $w := (y - h)/c$ satisfies an irreducible equation over A :

$$w^p + \dots + pc^{1-p}h^{p-1}w + g_0 = 0. \quad (6)$$

It follows that $w \in B$. If $\mu = p/(p-1)$ then Hensel's lemma, applied to the complete local ring A , would show that this equation is reducible. We conclude that $\mu < p/(p-1)$. Now Part (i) of the proposition is proved.

Let \bar{w} denote the image of w in $\bar{B} := B/\pi B$ and \bar{g}_0 the image of g_0 in $\bar{A} := A/\pi A = k[[t]]$. Then $m = \tilde{v}_0(\bar{g}_0)$. It is easy to see that for any choice of h such that $v_0(f - h^p) = \mu$ we have $\bar{g}_0 \notin \bar{A}^p$ (otherwise μ wouldn't be maximal). Moreover, choosing a different h results in adding to \bar{g}_0 an element of \bar{A}^p . We may therefore assume that $(p, m) = 1$. Furthermore, m is now the maximal possible value for $\tilde{v}_0(\bar{g}_0)$. It follows that (μ, m) is the maximal possible value for $v_\eta(f - h^p)$. This proves the first claim of the proposition and Part (ii), except for the statement that $m > 1$.

From (6) we see that \bar{w} satisfies the equation

$$\bar{w}^p = \bar{g}_0,$$

which is irreducible because $\bar{g}_0 \notin \bar{A}^p$. We conclude that \tilde{v}_0 is the unique extension of v_0 to $\text{Frac}(B)$, with residue field extension purely inseparable of degree p . Using Serre's Normality Criterion (as in Lemma 2.1) we also see that $A[w]$ is normal and hence $B = A[w]$. This proves (iii). Moreover, if $m = 1$ then we would have $B = R[[w]]$, i.e. Y was an open disk. This contradicts our assumption. Now the proof of the proposition is complete. \square

We continue to use the notation (μ, m) from the above proposition. We note that the values of μ and m do not change if we replace K by any finite extension. An element $h \in A$ such that $v_\eta(f - h^p) = (\mu, m)$ is called a *best approximation* of f (with respect to v_η) (cf. [2], §2.2.1).

Lemma 4.3 (p -Taylor expansion) *Let $n \geq 1$ be given, and set*

$$\nu_n := 1 + 1/p + \dots + 1/p^n.$$

After replacing K by a finite extension, there exists a parameter t for the disk X and an element $h \in A = R[[t]]$ such that

$$f - h^p = \sum_{j=1}^{\infty} a_j t^j$$

with $a_j \in \mathfrak{m}_R$ and

$$v(a_{pj}) \geq \nu_n, \quad (7)$$

for all j .

Following [16], we call (t, h, a_j) a p -Taylor expansion of f of level n .

Proof: See [2], Proposition 2.12. \square

Since $\nu_n \rightarrow p/(p-1)$ for $n \rightarrow \infty$, we may choose n such that

$$\nu_n > \frac{p}{p-1} - \frac{p/(p-1) - \mu}{m}. \quad (8)$$

Let (t, h, a_j) be a p -Taylor expansion of f of level n . Since $\nu_n > \mu \geq v_0(f - h^p)$, it follows from (8) that the minimum of the valuations $v(a_j)$ occurs for an index j which is prime to p . By inspection of the proof of Proposition 4.2 one concludes that $v_\eta(f - h^p) = (\mu, m)$, i.e. a p -Taylor expansion of f yields a best approximation (see [2], Corollary 2.16).

We define

$$\rho := \min \left\{ \frac{p/(p-1) - \mu}{m}, \frac{v(a_j) - \mu}{m-j} \mid 1 \leq j < m \right\}.$$

Note that $\mu + m\rho \leq p/(p-1)$. If equality holds, we set $k := 0$. Otherwise, we let k denote the smallest index such that $1 \leq k < m$ and $\rho = (v(a_k) - \mu)/(m - k)$. Then by definition the Newton polygon of $f - h^p$ has a line segment of slope $-\rho$ over the interval $[k, \dots, m]$. Moreover, it follows from (7) and (8) that $(k, p) = 1$.

One can check that ρ and k do not depend on the choice of h (see [2], Proposition 2.28). However, ρ and k may depend on the choice of the parameter t !

Lemma 4.4 *After a finite extension of K , there exists a p -Taylor expansion (t, h, a_j) of level n such that $k \neq 1$.*

Proof: See [2], Proposition 2.31. The idea is to use a ‘generic’ p -Taylor expansion

$$f(t + T) - H^p = \sum_{j=1}^{\infty} A_j t^j,$$

where the A_j and the t -coefficients of H are algebraic functions in T . One has to show that, after a finite extension of K , there is a point $T = \xi$ such that $A_1(\xi) = 0$. Replacing the parameter t by $t' := t - \xi$ then gives a p -Taylor expansion (t', h', a'_j) with $a'_1 = 0$. With respect to this p -Taylor expansion we have $k \neq 1$. \square

Now the following proposition completes the proof of Proposition 4.1 in the special case considered in this subsection.

Proposition 4.5 *Let (t, h, a_j) be a p -Taylor expansion of f such that $k \neq 1$ (notation as above). Then $D := X(|t| \leq \rho)$ is the minimal exhausting disk with respect to ϕ .*

Proof: Let $U := X \setminus D$. Choose an element $b \in R$ such that $v(b) = \rho$ and set $t_1 := t/b$, $s_1 := t_1^{-1}$ (in general, this requires again a finite extension of K). Then

$$A' := \Gamma(D, \mathcal{O}_X^\circ) = R\{\{t_1\}\}$$

(the ring of convergent power series over R) and $B' := \Gamma(\phi^{-1}(D), \mathcal{O}_Y^\circ)$ is the integral closure of A' in the extension of fraction fields given by $y^p = f$. Similarly,

$$A'' := \Gamma(U, \mathcal{O}_X^\circ) = R[[t, s_1 \mid ts_1 = b]],$$

and $B'' := \Gamma(\phi^{-1}(U), \mathcal{O}_Y^\circ)$ is the integral closure of A'' .

By construction, the Newton polygon of the power series $f - h^p = \sum_{j \geq 1} a_j t^j$ has a line segment of slope $-\rho$ over the interval $[k, m]$ and, moreover, $-\rho$ is the largest negative slope that occurs. It follows that we can write $f - h^p$, as an element of the ring A'' , in the form

$$f - h^p = c_1^p t^m u_1,$$

where $c_1 \in R$ is an element of valuation μ/p and u_1 is a principal unit. It follows that the element $w_1 := (y - h)/c_1$ is an element of B'' satisfying an irreducible equation over A'' of the form

$$w_1^p + \dots + p c_1^{1-p} h^{p-1} = t^m u_1.$$

Now it follows from [2], Lemma 1.31, that $\phi^{-1}(U)$ is an open annulus. We have shown that D is exhausting.

To show that D is the minimal disk with this property we shall provide an explicit description of the canonical reduction W° of $\phi^{-1}(D)$ (we freely use the notation set up in §3.5). Set $\lambda := \mu + m\rho$. Then $\lambda \leq p/(p-1)$ and we can write

$$f - h^p = c_2^p g,$$

with $g \in A''$ and where $c_2 \in R$ is chosen such that $v(c_2) = \lambda/p$. Then the element $w := (y - h)/c_2$ satisfies the integral equation

$$w^p + \dots + p c_2^{1-p} h^{p-1} = g$$

over A'' and therefore $w \in B''$. Let $\bar{g} \in \bar{A}'' := A''/\pi A'' = k[t_1]$ (resp. $\bar{w} \in \bar{B}'' := B''/\pi B''$) denote the image of g (resp. of w).

Suppose first that $\lambda = p/(p-1)$. Then \bar{w} satisfies an Artin-Schreier equation

$$\bar{w}^p + \bar{c}\bar{w} = \bar{g}.$$

Moreover, $\bar{g} \in k[t_1]$ is a polynomial in t_1 of degree $m > 1$, $(m, p) = 1$. It follows that $W^\circ = \text{Spec } k[t_1, \bar{w}]$ is a smooth and connected affine curve of genus

$g = (p-1)(m-1)/2 > 0$. Using Lemma 3.11 (ii) we conclude that D is the minimal exhausting disk.

Now suppose that $\lambda < p/(p-1)$. Then \bar{w} satisfies the equation

$$\bar{w}^p = \bar{g},$$

where $\bar{g} = \bar{g}_k t_1^k + \dots \bar{g}_m t_1^m$ is a polynomial of degree m , divisible by t_1^k . The affine curve $W^\circ = \text{Spec } k[t_1, \bar{w}]$ is irreducible and has a unibranched singularity precisely over each point of $Z^\circ = \text{Spec } k[t_1]$ where the differential $d\bar{g}$ has a zero. Since $1 < k < m$ and $(p, k) = (p, m) = 1$ it follows that W° has at least two singular points. Using Lemma 3.11 (ii) again we conclude that D is the minimal exhausting disk. \square

4.5 We can now prove the general case of Proposition 4.1. By the result of the previous subsections we may assume that G has a proper normal subgroup $H \triangleleft G$. Set $Z := Y/H \rightarrow X$ and consider the factorization

$$Y \xrightarrow{H} Z \xrightarrow{G/H} X$$

of ϕ into regular Galois subcovers.

Suppose Z is not a disk. Then by the induction hypothesis there exists a maximal boundary component $X_1 \subset X$ which is separating with respect to $Z \rightarrow X$. If Z is an open disk then we set $X_1 := X$.

Choose a connected component Y_1 of the inverse image of X_1 in Y . Let $G_1 \subset G$ denote the stabilizer of Y_1 in G and set $H_1 := G_1 \cap H$. Note that $H_1 \triangleleft G_1$ is a normal subgroup. We can identify the quotient $Z_1 := Y_1/H_1$ (resp. Y_1/G_1) with a connected component of the inverse image of X_1 in Z (resp. with X_1). By construction, Z_1 is either an open disk (and then $Z_1 = Z$) or an open annulus. Applying the induction hypothesis once more to the H_1 -cover $Y_1 \rightarrow Z_1$ we obtain a maximal separating boundary domain $Z_2 \subset Z_1$ with respect to $Y_1 \rightarrow Z_1$.

We claim that the subset $Z_2 \subset Z_1$ is fixed by the action of G_1/H_1 . Indeed, any element $g \in G_1$ induces an isomorphism of the cover $Y_1 \rightarrow Z_1$. It follows that $g(Z_2) \subset Z_1$ is also a maximal separating boundary domain with respect to $Y_1 \rightarrow Z_1$. Uniqueness shows that $g(Z_2) = Z_2$.

The quotient $X_2 := Z_2/(G_1/H_1)$ can be identified with a boundary domain of X . We claim that X_2 is separating with respect to the cover $\phi : Y \rightarrow X$, and is maximal with respect to this property. Indeed, let Y_2 denote a connected component of the inverse image of X_2 in Y . We may assume that Y_2 is contained in Y_1 . Then Y_2 is also a connected component of the inverse image of $Z_2 \subset Z_1$ in Y_1 . By the choice of Z_2 this means that Y_2 is an open annulus. This shows that X_2 is separating with respect to the cover ϕ . The maximality of X_2 is proved in a similar manner. This completes the proof of Proposition 4.1. \square

5 The nonsolvable case

5.1 We return to the situation considered in §3.4: we are given a regular G -Galois cover $\phi : Y \rightarrow X$, where X is an open disk and Y is an open analytic curve which is *not* an open disk. Our goal is to show that the set \mathcal{D} of all affinoid disks $D \subset X$ which are exhausting with respect to ϕ has a unique minimal element (Theorem 3.9). In view of Proposition 4.1 we may assume that the group G is *not* solvable.

5.2 We let $\mathcal{X} = \mathrm{Spf}(A)$ and $\mathcal{Y} = \mathrm{Spf}(B)$ denote the canonical formal models. We let $\eta \in \partial X$ denote the unique boundary point of X . We let $k[\eta] \subset k(\eta)$ denote the valuation ring of the residue field of η . In this section, we consider η as a morphism of formal schemes $\eta : \mathrm{Spf}(k[\eta]) \rightarrow \mathcal{X}$.

Let $D \subset X$ be an affinoid disk. By a local variant of the procedure described in §3.6, D gives rise to a diagram of formal R -schemes

$$\begin{array}{ccc} \mathcal{Y}' & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{X}' & \longrightarrow & \mathcal{X}, \end{array}$$

as follows. Let $t \in A$ be a parameter and $\epsilon \in |K^\times|$ such that $D = X(|t| \leq \epsilon)$. Let $\mathcal{X}' \rightarrow \mathcal{X}$ be the formal blowup of the ideal $I := (t, a) \triangleleft A$. Let $Z \subset \mathcal{X}'$ be the exceptional fiber (it is equal to the reduced subscheme $(\mathcal{X}')^{\mathrm{red}}$, and it is isomorphic to \mathbb{P}_k^1). The morphism $\xi : \mathrm{Spf}(k[\eta]) \rightarrow \mathcal{X}$ lifts uniquely to a morphism $\xi' : \mathrm{Spf}(k[\eta]) \rightarrow \mathcal{X}'$. Let $z \in Z$ denote the image of ξ' and $Z^\circ := Z \setminus \{z\}$. Then

$$D =]Z^\circ[_{\mathcal{X}'}.$$

Let \mathcal{Y}' be the normalization of the formal scheme \mathcal{X}' in \mathcal{Y} (see ??). We call \mathcal{Y}' the formal model of Y induced by D . Let $W := (\mathcal{Y}')^{\mathrm{red}}$ denote the reduced subscheme. Note that W is a connected projective k -curve. The canonical morphism $\mathcal{Y}' \rightarrow \mathcal{X}'$ restricts to a finite map $W \rightarrow Z$. Let $\partial W \subset W$ denote the inverse image of z and $W^\circ := W \setminus \partial W$. We have

$$\phi^{-1}(D) =]W^\circ[_{\mathcal{Y}'}.$$

It follows that D is exhausting with respect to ϕ if and only if the residue classes $]w[_{\mathcal{Y}'}$ are open annuli, for all $w \in \partial W$. Actually, since G acts transitively on the set ∂W , it suffices that this holds for one $w \in \partial W$.

Given a boundary point $\xi \in \partial Y$ the morphism $\xi : \mathrm{Spf}(k[\xi]) \rightarrow \mathcal{Y}$ lifts uniquely to $\xi' : \mathrm{Spf}(k[\xi]) \rightarrow \mathcal{Y}'$, and the image $\bar{\xi} \in W$ of ξ' lies in ∂W . We obtain a surjective G -equivariant map

$$\partial Y \rightarrow \partial W, \tag{9}$$

the local analog of the map (5). If D is exhausting, then this map is a bijection.

5.3 We fix a boundary point $\xi \in \partial Y$. Let $D \subset X$ be either an affinoid disk or the empty set. Then we let $V(D)$ denote the connected component of $\phi^{-1}(X \setminus D)$ which ‘contains’ ξ . This means the following. If $D = \emptyset$ then $V(D)$ is equal to Y . On the other hand, if D is an affinoid disk then $V(D) :=]w[_{Y'_R}$, where $w \in \partial W$ is the image of ξ under the map (9). Note that $D \in \mathcal{D}$ if and only if $V(D)$ is an open annulus.

We let $G(D) \subset G$ denote the stabilizer in G of the component $V(D)$. Note that $V(D)/G(D) \cong X \setminus D$.

Lemma 5.1 *For $D \in \mathcal{D}$ the following holds.*

- (i) *The group $G(D)$ is equal to the stabilizer G_ξ in G of the boundary point ξ and is solvable.*
- (ii) *Let $D' \subset D$ be a subset which is either empty or an affinoid disk strictly contained in D . Then $V(D') \setminus \phi^{-1}(D)$ is absolutely connected. In particular, $\phi^{-1}(D)$ is absolutely connected.*

Proof: Since $D \in \mathcal{D}$, the map (9) is a bijection. The equality $G(D) = G_\xi$ follows immediately. Moreover, $G_\xi \subset G = \text{Gal}(\text{Frac}(B)/\text{Frac}(A))$ is the decomposition group of the discrete valuation on $\text{Frac}(B)$ corresponding to ξ . We obtain a short exact sequence

$$1 \rightarrow I_\xi \rightarrow G_\xi \rightarrow \text{Gal}(k(\xi)/k(\eta)) \rightarrow 1,$$

where I_ξ is the inertia group of ξ . The residue field extension $k(\xi)/k(\eta)$ is a finite extension of complete discrete valued fields with algebraic residue field. So $\text{Gal}(k(\xi)/k(\eta))$ is also an inertia group and hence solvable. We conclude that G_ξ is solvable.

For the proof of (ii) we first assume that $D' = \emptyset$, and we use the notation introduced in §5.2. We have already noted that W is connected. Since $D \in \mathcal{D}$, the subset ∂W consists of smooth points of W . It follows that the complement $W^\circ = W \setminus \partial W$ is still connected. Now Remark 3.1 (iv),(v) shows that

$$\phi^{-1}(D) =]W^\circ[_{Y'}$$

is absolutely connected, proving (ii) if $D = \emptyset$. The proof in the case $D' \neq \emptyset$ is similar and left to the reader. \square

5.4 Let X^{an} denote the Berkovich analytic space associated to X , see [3]. As a set, X^{an} consists of all continuous multiplicative seminorms $|\cdot|_x : A \rightarrow \mathbb{R}_{\geq 0}$ bounded by 1 which extend the standard valuation $|\cdot|$ on R . To each point $|\cdot|_x \in X^{\text{an}}$ we can associate its residue field $\mathcal{H}(x)$, which is defined as the completion of the fraction field of $A/\text{Ker}(|\cdot|_x)$. By construction, $\mathcal{H}(x)/K$ is an extension of complete valued fields. We let $\widetilde{\mathcal{H}(x)}$ denote the residue field of $\mathcal{H}(x)$.

Any point $x \in X$ gives rise to a point X^{an} by the formula $|f|_x := |f(x)|$. We may thus consider X as a subset of X^{an} (called the set of *classical points*). Classical points are characterized by the property that the extension $\mathcal{H}(x)/K$ is finite.

In order to have a uniform and suggestive notation, we shall write $x \in X^{\text{an}}$ instead of $|\cdot|_x \in X^{\text{an}}$ and $|f(x)|$ instead of $|f|_x$, for arbitrary points on X^{an} . For instance, to any closed disk $D \subset X$ (affinoid or not) we can associate a point $x_D \in X^{\text{an}}$ by setting

$$|f(x_D)| := \max_{x' \in D} |f(x')|.$$

If D is affinoid, then the residue field $\widetilde{\mathcal{H}(x)}$ can be identified with the function field of the canonical reduction of D . In particular, $\widetilde{\mathcal{H}(x)}/k$ has transcendence degree one. Otherwise, $\widetilde{\mathcal{H}(x)} = k$. See [3], §1.4.4.

The space X^{an} carries a natural topology which makes it a locally compact Hausdorff space. If $D \in X$ is an affinoid disk, then $D^{\text{an}} \subset X^{\text{an}}$ is a compact subset. It follows that the limit

$$x := \lim_{D \in \mathcal{D}} x_D \in X^{\text{an}}$$

exists. In fact, it is easy to see that for all $f \in A$ we have

$$|f(x)| = \inf_{D \in \mathcal{D}} |f(x_D)|.$$

Proposition 5.2 *Let $D_0 := \bigcap_{D \in \mathcal{D}} D$. Exactly one of the following cases occurs.*

- (1) *The limit point x is a classical point, and $D_0 = \{x\}$.*
- (2) *The set D_0 is an affinoid disk, and $x = x_{D_0}$.*
- (3) *The set D_0 is a closed disk which is not affinoid, and $x = x_{D_0}$.*
- (4) *The set D_0 is empty.*

In Case (1), (3) and (4) we have $\widetilde{\mathcal{H}(x)} = k$.

Proof: This follows from the classification of points on $(\mathbb{A}_K^1)^{\text{an}}$ in [3], §1.4.4. \square

Lemma 5.3 *Assume that we are in Case (1), (3) or (4) of Proposition 5.2. Then for any $y \in \phi^{-1}(x)$, the stabilizer $G_y \subset G$ of y is solvable.*

Proof: Let $\mathcal{O}_{X^{\text{an}}, x}$ denote the local ring of the point x on the analytic K -space X^{an} . By [4], Theorem 2.1.5, $\mathcal{O}_{X^{\text{an}}, x}$ is a henselian local ring. Moreover, by *loc.cit.*, Lemma 2.1.6, we have a decomposition

$$\phi_*(\mathcal{O}_{Y^{\text{an}}})_x = \prod_{i=1}^n \mathcal{O}_{Y^{\text{an}}, y_i},$$

where $\phi^{-1}(x) = \{y_1, \dots, y_n\}$. Therefore, the extension $\mathcal{O}_{Y^{\text{an}},y}/\mathcal{O}_{X^{\text{an}},x}$ is a Galois extension of henselian local rings with Galois group G_y . It follows that G_y sits in a short exact sequence

$$1 \rightarrow I_y \rightarrow G_y \rightarrow \text{Gal}(\kappa(y)/\kappa(x)) \rightarrow 1,$$

where $\kappa(x)$ and $\kappa(y)$ are the residue fields of $\mathcal{O}_{X^{\text{an}},x}$ and $\mathcal{O}_{Y^{\text{an}},y}$, respectively, and I_y is the inertia group. By [4], Proposition 2.4.3 and Proposition 2.4.4, I_y is solvable. It remains to show that $\text{Gal}(\kappa(y)/\kappa(x))$ is solvable. By [4], Theorem 2.3.3, $\kappa(x)$ is a henselian valued field (called *quasi-complete* in *loc.cit.*) whose completion is the field $\mathcal{H}(x)$. By Proposition 5.2, the residue field of $\mathcal{H}(x)$ (equal to the residue field of $\kappa(x)$) is equal to k , which is algebraically closed by assumption. Using again [4], Proposition 2.4.4 we conclude that $\text{Gal}(\kappa(y)/\kappa(x))$ and hence G_y is solvable. \square

Lemma 5.4 (i) *In Case (1) and (4) of Proposition 5.2, the affinoid disks D^{an} , $D \in \mathcal{D}$, form a neighborhood basis of x in X^{an} .*

(ii) *In Case (3), a neighborhood basis of x in X^{an} is given by the sets $(D \setminus D')^{\text{an}}$, where $D \in \mathcal{D}$ and D' is an affinoid disk contained in $D_0 = \cap_{D \in \mathcal{D}} D$.*

Proof: In Case (1), the statement in (i) is clear. Suppose we are in Case (4), i.e. $\cap_{D \in \mathcal{D}} D = \emptyset$. Any $f \in A$, seen as an analytic function on X^{an} , has finitely many zeroes, all of which are classical points. Therefore, there exists $D_1 \in \mathcal{D}$ such that $f|_{D^{\text{an}}}$ is an invertible analytic function on D^{an} , for all $D \in \mathcal{D}$ with $D \subset D_1$. Applying the maximum principle to $f|_{D^{\text{an}}}$ and $f^{-1}|_{D^{\text{an}}}$ we see that $|f(x)| = |f(x_D)|$. The statement in (i) now follows from the definition of the topology of X^{an} . The proof of (ii) is similar. \square

5.5 We can now finish the proof of Theorem 3.9. We first suppose that we are in Case (1) or (4) of Proposition 5.2. Let $\phi^{-1}(x) = \{y_1, \dots, y_n\}$ be the fiber above x . Then it follows from Lemma 5.4 and [4], proof of Theorem 2.1.5, that for all sufficiently small $D \in \mathcal{D}$ the inverse image $\phi^{-1}(D)$ decomposes into n disjoint affinoid neighborhoods of the points y_i . But $\phi^{-1}(D)$ is connected by Lemma 5.1 (ii), hence $n = 1$. By Lemma 5.3 this shows that the group G is solvable, contradicting our assumption. We conclude that Case (1) and (4) of Proposition 5.2 cannot occur.

We now suppose that we are in Case (2) of Proposition 5.2, i.e. $D_0 := \cap_{D \in \mathcal{D}} D$ is an affinoid disk. By Lemma 5.1 (i) we have

$$g(V(D)) \cap V(D) = \emptyset$$

for all $D \in \mathcal{D}$ and $g \in G \setminus G_\xi$. It follows easily that

$$V(D_0) = \cup_{D \in \mathcal{D}} V(D) \quad \text{and} \quad G(D_0) = G_\xi.$$

Consider the G_ξ -cover

$$V(D_0) \rightarrow X \setminus D_0.$$

If $D \subset D_0$ is an affinoid disk, then $D \in \mathcal{D}$ if and only if $U := X \setminus D$ is a separating boundary component for this cover, see §4. Since G_ξ is solvable by Lemma 5.1 (i), Proposition 4.1 shows that there exists a unique maximal separating boundary component. It follows that $D_0 \in \mathcal{D}$ is the unique maximal element of \mathcal{D} , i.e. Theorem 3.9 holds.

It remains to rule out Case (3) of Proposition 5.2. Using Lemma 5.4 (ii) we find an element $D \in \mathcal{D}$ and an affinoid disk $D' \subset D$ such that $x \in D^{\text{an}} \setminus (D')^{\text{an}}$ and such that $V(D')^{\text{an}} \setminus V(D)^{\text{an}}$ contains a unique point $y \in \phi^{-1}(x)$. By Lemma 5.3 this implies that the stabilizer of $V(D') \setminus V(D)$ in G is solvable. Together with Lemma 5.1 (ii) this shows that $G(D')$ is solvable. Applying Proposition 4.1 to the $G(D')$ -cover

$$V(D') \rightarrow X \setminus D'$$

and using a similar argument as in Case (2) we conclude that \mathcal{D} has a minimal element. But since this minimal element must be equal to the intersection $D_0 = \bigcap_{D \in \mathcal{D}} D$ which is *not* an affinoid disk in Case (3), we obtain a contradiction. Therefore, Case (3) of Proposition 5.2 cannot occur. This completes the proof of Theorem 3.9. \square

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